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A NOVEL APPROACH FOR THE SOLUTION OF DIRECT AND INVERSE PROBLEMS OF SOME EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract - It is proposed a new approach for solution of: 1) direct boundary value problems for the elliptic and parabolic partial differential equations; 2) coefficient inverse problems for the Laplace equation. The approach is based on the, proposed by author, General Ray Principle. It leads to new *GR*-method that uses the explicit formulas with fast inversion of the Radon transformation. The case of the domain with the complex geometry is considered. *GR*-method is realized by fast algorithms and MATLAB software, whose quality is justified by numerical experiments. Application to Electrical Tomography is presented.

1. INTRODUCTION

There are two main approaches for solving boundary value problems for partial differential equations in analytical form: Fourier decomposition and the Green function method. The numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. All methods and algorithms constructed on the bases of these approaches have some difficulties in realization for the complex geometrical form of the domain Ω. The Green function method is the explicit one [1], but for arbitrary coefficients of equations it is difficult to construct the Green function even for the simple geometry of Ω. Numerical approaches lead to solving systems of linear algebraic equations [2] that require a lot of computer time and memory. Hence, the development of new fast algorithms for solution of the problems under investigation is very actual.

We consider here a new approach for the solution of direct and inverse problems on the base of the General Ray Principle (*GRP*), proposed by the author in [3], [4] for the stationary waves field. *GRP* leads to explicit analytical formulas (*GR*-method) and fast algorithms, proposed firstly in [3] and developed numerically in [4] – [6] for the Dirichlet boundary value problem in an arbitrary simply connected star domain Ω with continuous contour Γ. Here we extend the proposed approach to more general type of equations and domains Ω for direct problems. We describe also the *GR*-method for the solution of the coefficient inverse problems for the elliptic equation of the Laplace type and its application to Electrical Tomography.

2. STATEMENT OF DIRECT BOUNDARY PROBLEMS FOR ELLIPTIC AND PARABOLIC EQUATIONS

Here we consider the Dirichlet or Neumann boundary value problems for elliptic equations, namely

$$\left\{\begin{array}{l} \nabla\left(\varepsilon\left(x,y\right)\nabla u\left(x,y\right)\right)+k^2u(x,y)=\psi\left(x,y\right), \quad (x,y)\in\Omega; \\ \quad u\left(x,y\right)=f, \quad (x,y)\in\Gamma; \\ or \quad \dfrac{\partial u}{\partial n}=g \quad (x,y)\in\Gamma; \end{array}\right\} \tag{1}$$

with respect to the function $u(x,y)$ inside the plane domain Ω with a boundary Γ. Here $f(x,y)$ or $g(x,y)$ are given functions for $(x,y)\in\Gamma$, k is a real number, $\varepsilon(x,y)>0$. If $k=0$, $\psi(x,y)=0$, we have the Laplace equation written in the divergent form. The case $\psi(x,y)\neq 0$ corresponds to Poisson equation. For $k\neq 0$, $\varepsilon(x,y)=1$ we have the Helmholtz equation. The problem (1) describes the distribution of the "potential" function $u(x,y)$ for any field of stationary waves, which can be interpreted as electrostatic, elastic or optic field [1], [2].

We consider the boundary value problem for the parabolic equation in the form:

$$\left\{\begin{array}{l} u_t(t,x)=u_{xx}(t,x), \quad -1<x<1, \ t>0; \\ \quad u\left(0,x\right)=f_0(x), \quad -1\le x\le 1; \\ u\left(t,-1\right)=f_{-1}(t), \quad u\left(t,1\right)=f_{+1}(t), \quad t\ge 0; \end{array}\right\} \tag{2}$$

The specific element of this traditional statement is the unbounded character of the domain Ω and the corresponding boundary Γ in range of variables *t* and *x*: $\Omega = \overline{\Omega} \equiv [0, \infty) \times [-1,1]$. We will suppose the finiteness of the solution in the domain $\overline{\Omega}$.

3. LOCAL RAY PROPERTY AND GENERAL RAY METHOD

In [3], [4] the General Ray (GR) principle was proposed that gives no traditional formalization of mathematical models for considered physical field and corresponding direct and inverse problems. GR-principle consists in the next main assumptions:

1) the physical field can be simulated mathematically by the superposition of plane vector fields $\vec{V}(l)$, each of them is parallel to the direction along some straight line l, at the superposition corresponds to all lines l;

2) the field $\vec{V}(l)$ is characterized by some function $u(x, y)$;

3) we know some characteristics such as values of function $u(x, y)$ and/or flow of the vector $\vec{V}(l)$ in any boundary point $P_0 = (x^0, y^0)$ of the domain.

Application of the of the GR Principle to the problems under investigation means to construct an analogue of eqns. (1), (2) describing the distribution of the function $u(x, y)$ and $u(t, x)$ along of "General Local Rays", which are presented by some straight line l with parameterization due a parameter τ : $x = p\cos\varphi - \tau\sin\varphi$, $y = p\sin\varphi + \tau\cos\varphi$, in the case of the elliptic equation, or with parameterization: $x = p\cos\varphi - \tau\sin\varphi$, $t = p\sin\varphi + \tau\cos\varphi$ in the case of the parabolic equation. Here $|p|$ is a length of the perpendicular from the centre of coordinates to the line l, $\varphi \in [0, \pi]$ is the angle between the axis *x* and this perpendicular. Using this parameterization, we shall define functions $u(x, y)$ (and $u(t, x)$ for eqn. (2)), $\varepsilon(x, y)$, $f(x, y)$, $g(x, y)$, $\psi(x, y)$ at $(x, y) \in l$ or at $(t, x) \in l$ for fixed p, φ as functions $u(\tau)$, $\varepsilon(\tau)$, $f(\tau)$, $g(\tau)$, $\psi(\tau)$ of variable τ. We suppose that the domain Ω is a convex one. Let us define for every fixed p and φ the functions $u_0(p,\varphi) = u(\tau_0)$, $u_1(p,\varphi) = u(\tau_1)$, for parameters τ_0, τ_1, which correspond to the points of the intersection of the line l and boundary of the domain.

Hence, the GR Principle leads to the assemblage (depending of p, φ) of ordinary differential equations:

$$\begin{aligned} &\left(\varepsilon(\tau) u_\tau(\tau)\right)_\tau + k^2 r^2(\varphi) u(\tau) = r^2(\varphi)\psi(\tau), \tau \in [\tau_0, \tau_1]; \\ &r^2(\varphi) = \sin^2\varphi\cos^2\varphi, \end{aligned} \tag{3}$$

as the local analog of the eqn. (1), and the corresponding analog of the eqn. (2):

$$\begin{aligned} &u_{\tau\tau}(\tau) - K u_\tau(\tau) = 0, \quad \tau \in [\tau_0, \tau_1], \\ &K = \frac{\sin^2\varphi}{\cos\varphi}, \quad \varphi \neq \frac{\pi}{2} \quad . \end{aligned} \tag{4}$$

Boundary conditions lead to the corresponding local boundary conditions for $u(\tau)$ at points τ_0, τ_1. We will designate the solution of the local problem (3) or (4) with such boundary conditions as $\overline{u}(\tau)$. For standard domains such as a circle or rectangular $\overline{\Omega}$ it is simple to calculate τ_0, τ_1 and so the functions $u_0(p,\varphi)$, $u_1(p,\varphi)$, using boundary functions f, g, or f_0, f_{-1}, f_{+1}, and then obtain the solution $\overline{u}(\tau)$ in explicit analytical or approximate form, using well known standard formulas and numerical methods for the solution of boundary value problems for ordinary differential equations.

Formulation of Local Ray property (*LRP*): *the adequate local description of the solution u on any straight line l (Local Ray) can be represented by the function $\overline{u}(\tau)$, so that the following final global formula for the solution of eqns (1) and (2) is true*

$$u(\xi, \eta) = R^{-1}\left[\int_{\tau_0}^{\tau_1} \overline{u}(\tau)\, d\tau\right], \quad (\xi, \eta) \in \Omega , \tag{5}$$

where R^{-1} is inverse Radon transform, (ξ,η) =(x, y) for the solution of elliptic problem (1), (ξ,η) =(t, x) for the solution of parabolic problem (2).

Formula (5) gives the explicit solution for a considered class of boundary value problems in arbitrary convex domains Ω with continuous contour Γ. Numerical realization of this formula we named the General Ray (*GR*) Method for the direct problems. We will concretise below formula (5) for particular cases of equations and demonstrate its validity by numerical examples.

We suppose that the function $u(x, y)$ and its first derivatives are localised in Ω, i.e. they are equal to zero outside Ω. We want to underline that under the assumption made we use the Radon transformation [14] in local imaging modality that implies validity of considerations presented below.

4. SOLUTION OF DIRECT BOUNDARY VALUE PROBLEMS

For k=0 , $\psi(x, y)$=0, in the case of the Laplace equation with variable coefficients, we obtain the mentioned analogue of eqn. (1) on the line l for every fixed p and φ given by the following ordinary differential equation

$$(\varepsilon(\tau)\, u_\tau(\tau))_\tau = 0 . \tag{6}$$

We introduce the functions

$$\gamma(\tau) = 1/\varepsilon(\tau); \quad k_0(\tau) = \int_{\tau_0}^{\tau_1} \gamma(\xi)\, d\xi;$$
$$K_0(p,\varphi) = \int_{\tau_0}^{\tau_1} k_0(\xi) d\xi; \quad K_1(p,\varphi) = k_0(\tau_1). \tag{7}$$

Then, integrating twice eqn. (6), we obtain for the solution of the Dirichlet problem the following formula

$$u(x, y) = R^{-1}[u_0(p,\varphi)(\tau_1 - \tau_0) + \frac{u_1(p,\varphi) - u_0(p,\varphi)}{K_1(p,\varphi)} K_0(p,\varphi)]. \tag{8}$$

For the case $\varepsilon(x, y) = 1$ we have the more simple formula

$$u(x, y) = R^{-1}\left[\frac{u_1(p,\varphi) + u_0(p,\varphi)}{2}(\tau_1 - \tau_0)\right] . \tag{9}$$

For the solution of the Neumann boundary value problem we present here the formula, corresponding to the domain of the unit circle and $\varepsilon(x, y) = 1$:

$$u(x, y) = R^{-1}\left[(g_1(p,\varphi) + g_0(p,\varphi))(1-p^2)^{1/2}\right] + C, \tag{10}$$

where C is arbitrary constant, functions $g_i(p,\varphi),\quad i = 0,1$, correspond to the Neumann boundary condition function $g(x, y)$ in (1), calculated at the boundary points τ_0, τ_1. The unique solution can be obtained from function (10) by additional interpolation condition in one point on the boundary.

The Dirichlet boundary value problem for the Poisson equation corresponds to $k=0$, $\psi(x, y)\neq 0$. The main formula for its solution is the following

$$u(x,y) = R^{-1}\left[\frac{u_1(p,\varphi)+u_0(p,\varphi)}{2}(t_1 - t_0)\right] + R^{-1}\left[r^2(\varphi)\left\{\psi_3(t_1) - \psi_3(t_0) - \frac{\psi_2(t_0)+\psi_2(t_1)}{2}(t_1 - t_0)\right\}\right] \tag{11}$$

where $\psi_2(t)$, $\psi_3(t)$ are the second and the third primitive functions of $\psi(t)$.

For $k \neq 0$, $\varepsilon(x, y) = 1$ we have the Helmholtz equation. For the no resonance case, when $\bar{k} \equiv k\, r(\varphi)\sqrt{(1-p^2)} \neq \pi(1+2m)$, m =0,±1, ±2,…, the solution of the Dirichlet problem is given by formula:

$$u(x, y) = R^{-1}\left[\frac{u_1(p,\varphi) + u_0(p,\varphi)}{kr(\varphi)} \tan(\bar{k}/2)\right] . \tag{12}$$

The resonance case is just under the author investigation.

The main formula for the solution of the boundary value problem (2) for parabolic equations is the following

$$u(t,x)=R^{-1}\left[\frac{u_1(p,\varphi)-u_0(p,\varphi)}{K}+u_0(p,\varphi)(\tau_1-\tau_0)\right]-R^{-1}\left[\frac{u_1(p,\varphi)-u_0(p,\varphi)}{e^{K(\tau_1-\tau_0)}-1}(\tau_1-\tau_0)\right]. \quad (13)$$

5. TRANSFORMATION OF DOMAINS WITH COMPLICATED GEOMETRY TO THE UNIT CIRCLE

In [4] - [6] it was proposed a reduction of the Dirichlet problem for the Laplace equation for an arbitrary simply connected star domain Ω with continuous contour Γ to the same problem on the unit circle. We make some change of variables, using equation for the curve Γ, which leads to the same problem with the standard Γ as the unit circle circumference. Let the continuous contour Γ of the plane simply connected star shaped domain Ω be represented as the curve, defined in the polar system of coordinates r,α by equation $r=r_0(\alpha)$, where r_0 is known function that does not vanish. The mentioned transformation of the domain Ω to the unit circle is the affine mapping, determined in the new polar coordinates $\tilde{r}$, $\tilde{\alpha}$ by the following formula

$$\tilde{\alpha}=\alpha, \quad \tilde{r}=r/r_0(\alpha), \quad (14)$$

which does not change the Laplace equation.

The mapping (14) does not require solution of any equations, does not include any bulky manipulation with complex variables. Hence, this transformation is realised by very fast algorithm, which is justified in [4], [5], [6] by numerical experiments for sufficiently complicated functions and domains.

This transformation is generalized for domains, compound of a finite number of the simply connected star domains. Some examples are presented below for the Laplace equation.

We generalized also the developed approach for the class of boundary value problems for elliptic eqns (1). We can reduce such problems to the similar ones on the unit circle using corresponding modifications of the coefficient k and the functions $g(x,y)$, $\psi(x,y)$.

6. RESULTS OF NUMERICAL EXPERIMENTS FOR DIRECT PROBLEMS

We have constructed the algorithmic and program realization of *GR*–method for various types of problems in MATLAB. We used the uniform discretization of variables $p\in[-1,1]$, $\varphi\in[0,\pi]$, so as for variables *t, x, y*, with *n* nodes. To calculate the inverse Radon transform for discrete data we constructed the original modification [7] of *iradon* program from MATLAB package. We made testes on mathematically simulated model examples with known exact functions $u(x,y)$, $\varepsilon(x,y)$, $f(x,y)$, $g(x,y)$, $\psi(x,y)$.

We present in Figure 1 some results for solution of the Dirichlet boundary value problem for Laplace equation on the well known standard simply connected star domain as "cross" - in graph a), and compound from two simply connected star domains "double cross" - in graphs b) – e).

We present below numerical experiments of solution for elliptic problems, which demonstrate the quality of the *GR*-method, for the case of the unit circle, because the case of more complicated domains Ω can be reduced to it. It is sufficiently to choose parameters τ_0,τ_1 for the unit circle circumference by formulas $\tau_{0,1}=\mp(1-p^2)^{1/2}$, and then calculate the functions $u_i(p,\varphi)=f(x^i,y^i)$, $x^i=p\cos\varphi-\tau_i\sin\varphi$, $y^i=p\sin\varphi+\tau_i\cos\varphi,\ i=0,1$.

Let us define as $u_n(x,y)$ the approximations obtained by formulas (8) - (13) for discrete case. We introduce the discrete relative medium estimations *rm* to demonstrate the quality of approximation:

$$rm(n)=\frac{1}{n^2}\frac{\sum_{i,j=1}^{n}|u_n(x_i,y_j)-u(x_i,y_j)|}{\max\limits_{(x_i,y_j)\in\Omega}|u(x_i,y_j)|}. \quad (15)$$

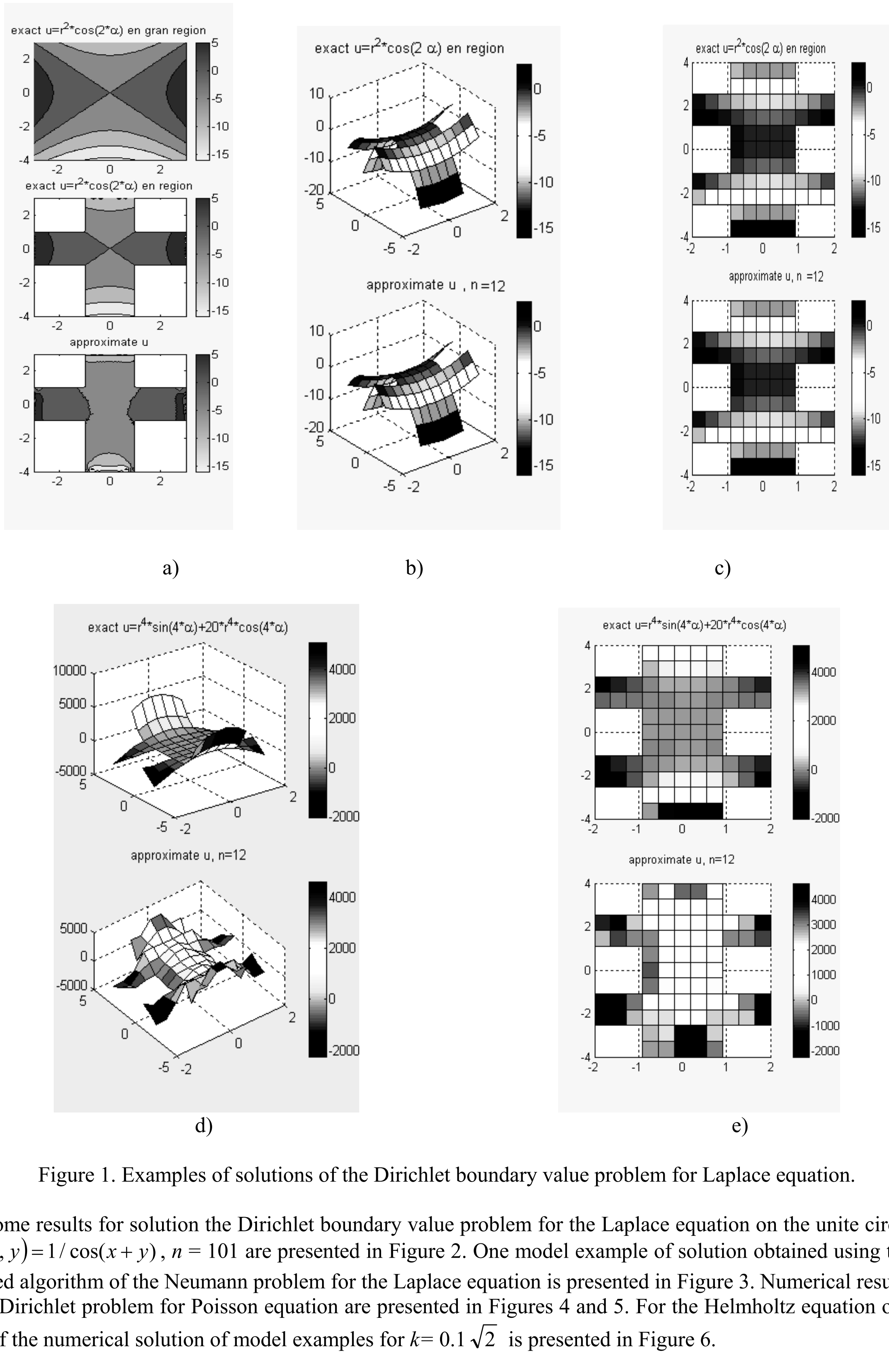

Figure 1. Examples of solutions of the Dirichlet boundary value problem for Laplace equation.

Some results for solution the Dirichlet boundary value problem for the Laplace equation on the unite circle for $\varepsilon(x, y) = 1/\cos(x + y)$, n = 101 are presented in Figure 2. One model example of solution obtained using the proposed algorithm of the Neumann problem for the Laplace equation is presented in Figure 3. Numerical results for the Dirichlet problem for Poisson equation are presented in Figures 4 and 5. For the Helmholtz equation one result of the numerical solution of model examples for k= $0.1\sqrt{2}$ is presented in Figure 6.

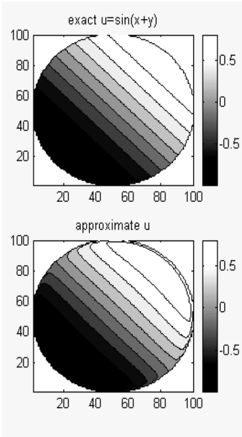

Figure 2. n=101; rm = 0.0452; $\varepsilon(x, y) = 1/\cos(x + y)$

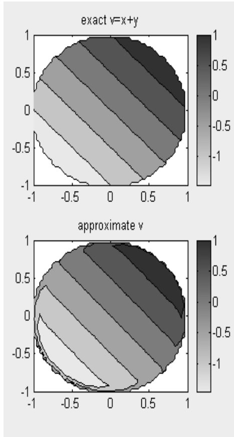

Figure 3. Solution of the Neumann problem.

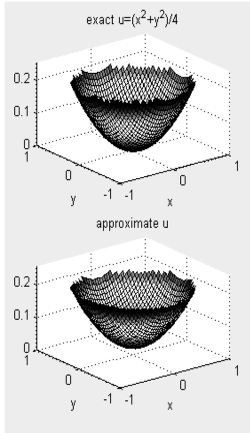

Figure 4. n=50; rm = 0.0265.

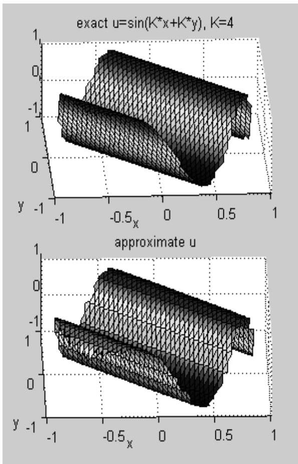

Figure 5. n=50, rm = 0.1403.

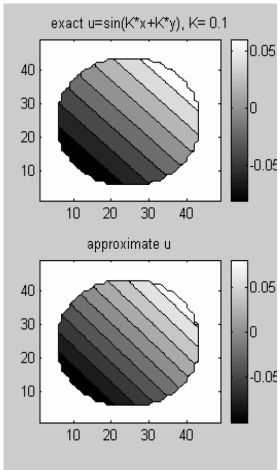

Figure 6. Solution of the Helmholtz equation.

Some examples of solution of the parabolic problem (2) are presented in Figures 7 – 9.

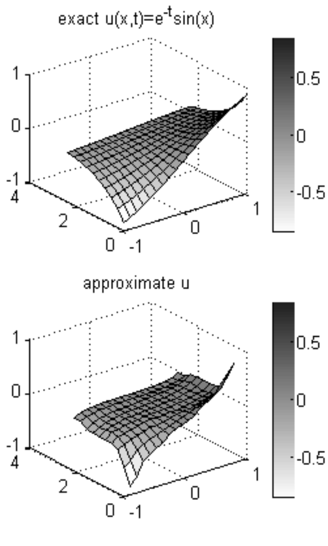

Figure 7.

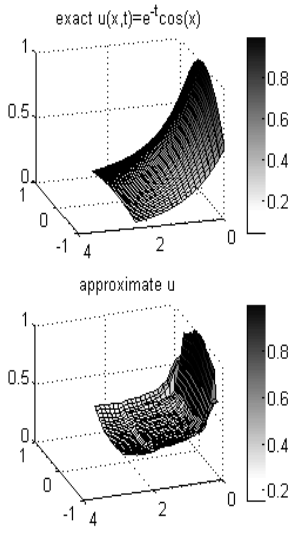

Figure 8.

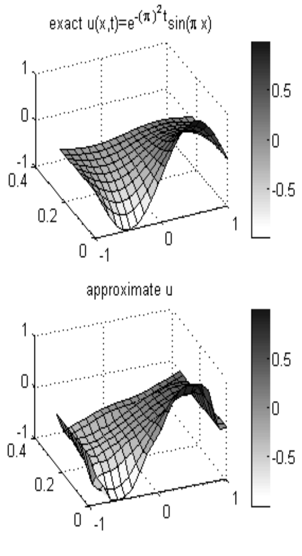

Figure 9.

7. STATEMENT OF COEFFICIENT INVERSE PROBLEM FOR THE LAPLACE TYPE EQUATION
Let us consider a coefficient inverse problem for the Laplace type eqn. (1) when $k=0$, $\psi(x,y)=0$, with respect to the function $\varepsilon(x,y)$, with functions $J_n(x,y)$, $u^0(x,y)$ given on the curve Γ and the following boundary conditions satisfied:

$$\varepsilon(x,y)\frac{\partial u(x,y)}{\partial n} = J_n(x,y), \quad (x,y)\in\Gamma \tag{16}$$

$$u(x,y) = u^0(x,y), \quad (x,y)\in\Gamma . \tag{17}$$

Here $\frac{\partial}{\partial n}$ is the normal derivative at the points of the boundary curve Γ. Traditional approach for solving this inverse problem leads to a nonlinear ill-posed problem [11].

We propose here another approach and statement that use *GR*–principle, i.e. we consider the field described by potential function $u(x,y)$ as the stream flow of “general rays". Let us describe one example to demonstrate reasons in favour of applicability of the GR–principle. For simplicity we put below Ω as the unit circle. In Figure 10 we present the corresponding model structure, which consists of two concentric circles: unit circle Ω as the homogeneous background with $\varepsilon(x,y)\equiv 1$ and one non-homogeneous element Ω_1 as the circle of radius $r<1$. Really, as it is analytically proved in [13] for the case of the electric field and is shown on the left part of Figure 10, the field has a small perturbation in the neighbourhood of Ω_1, caused by this non-homogeneity. Let us change the region Ω_1 by n rectangles with the long boards parallel to the line l and with short ones intersecting the internal circumference, as it is shown in the right part of Figure 10.

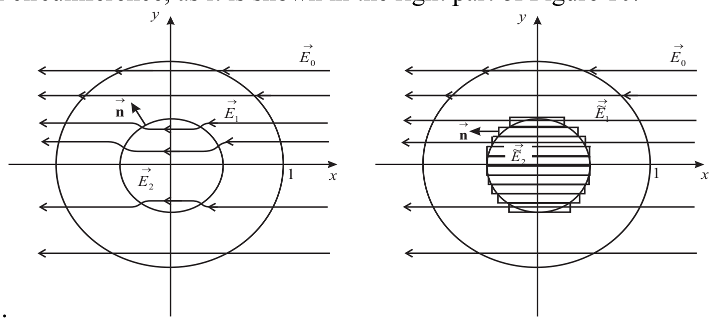

Figure 10. Interpretation of the domain approximated by rectangles
in justification of applicability of GR-principle.

Suppose that the values of the potential and induction at the points of this approximate boundary are equal to the values of the potential and induction of the external field. That is, we neglect the small perturbation of the external field in the neighbourhood of Ω_1, caused by this non-homogeneity. So, as it demonstrated in the right part of Figure 10, even in the neighbourhood of Ω_1, we can consider the lines of the electric field as straight lines.

The *GR*–principle gives us eqn. (3) . We consider also the following boundary conditions

$$\varepsilon(\tau_0)u_\tau(\tau_0) = J(p,\varphi) \tag{18}$$

$$u(-\tau_0) - u(\tau_0) = v(p,\varphi), \ \tau_0 = -(1-p^2)^{1/2} \tag{19}$$

for given functions $v(p,\varphi)$ and $J(p,\varphi)$. Equations (6), (18) and (19) constitute the basic mathematical model for the inverse problem of reconstructing the coefficient $\varepsilon(x,y)$.

Supposing that different components in the considered structure have the smooth distribution such that the functions $\varepsilon(\tau)u_\tau(\tau)$ and $u_\tau(\tau)$ are continuous and integrating twice the eqn. (6) with respect to τ we obtain for $\varepsilon(x,y)$ the following formula

$$\varepsilon(x,y) = 1 \big/ R^{-1}\left[\frac{v(p,\varphi)}{J(p,\varphi)}\right] \tag{20}$$

where R^{-1} is the inverse Radon transform operator. Formula (20) represents the Scanning General Ray method for the inverse problem. This formula can be generalised and applied also for structures with piecewise constant characteristics.

8. APPLICATION TO THE ELECTRICAL TOMOGRAPHY

Electrical Impedance Tomography (EIT) is the most developed approach in electrical tomography that includes the resistance (ERT) and capacitance (ECT) tomography [15]. We propose here another variant of the electrical tomography, when the external electromagnetic field $\vec{V}(l)$ is produced by active electrodes, located outside of Ω. It initiates some distribution of the electric potential inside the domain Ω. We propose that measurements of necessary values would be realized on the boundary curve Γ with another, no active electrodes. Let the external field in scanning parallel beam scheme be electrostatic homogeneous in the direction orthogonal to the line l. Then we can use the mathematical model (6), (18) and (19) for this kind of tomography, which, in the particular case of symmetrical structures, was considered in [10] and is called "*GR*-tomography".

Formula (20) demonstrates the difference between the proposed approach and the electric resistance or capacitance tomography, in which only the normal component $J(p,\varphi)$ is used. In ERT the given normal component $J(p,\varphi)$ means the value of the normal current in one point of the intersection of the line l and boundary. Function $v(p,\varphi)$ is measured. In ECT the function $v(p,\varphi)$ is given and the normal component of $J(p,\varphi)$ is related with measured mutual capacitances.

In the proposed scheme the function $v(p,\varphi)$ might be measured and $J(p,\varphi)$ can be calculated approximately, using known characteristics of the external field. For example, if the right hand sides in the boundary conditions (16) and (17) are given for every scanning angle φ, we can obtain, in accordance with a parallel beam measurement scheme, the data $v(p,\varphi)$ and $J(p,\varphi)$ for $p\in[-1,1]$, $\varphi\in[-\pi/2,\pi/2]$.

In both ERT and ECT schemes the electric field is produced by the same electrodes that serve as measuring elements, i.e. the electrodes are active. The mutual influence of electrodes is the cause of impossibility to use a greater number of electrodes and obtain sufficiently large number of significant measurements. It is very important that in proposed scheme measuring electrodes are not active and serve only for acquisition of data $v(p,\varphi)$ and $J(p,\varphi)$. Therefore, the proposed approach gives, in principle, the possibility to use a large number of electrodes and measurements of the input values of functions $v(p,\varphi)$ and $J(p,\varphi)$ and reconstruct the desired image more perfectly.

We have constructed the numerical realization of formula (20) that we call "scanning *GR*–algorithm".
This algorithm does not require solving any equation, because the Radon transform can be inverted by a fast algorithm using discrete FFT algorithm, which is realised in MATLAB.

Analysis of formulas for inverse Radon transformation shows that its instability for discrete noised data is equivalent to the instability of the problem of the numerical differentiation of the noised function $\bar{v}(p,\varphi)$ with respect to the variable p. The regularization of the inversion of Radon transform was constructed by author on the base of the Recursive Smoothing (RS) by splines [8]. Theoretical and numerical justification of the regularization properties of this type of smoothing are presented in [8] - [10], [12]. If for structures with piecewise constant characteristics the set $\hat{\varepsilon}=\{\varepsilon_0,\varepsilon_1,\varepsilon_2\}$ of the known values ε_i is given, then the algorithm includes also the projection of the pre-reconstructed data to the set $\hat{\varepsilon}$ with respect to the absolute or relative criterion [10].

9. NUMERICAL EXPERIMENTS FOR INVERSE PROBLEM

We tested scanning *GR*-algorithm on mathematically simulated model examples. The experiments presented correspond to piecewise constant structure inside the unit circle and the external field such that $J(p,\varphi)\equiv 1$. In these model examples we use the mentioned interpretation of approximate form of the boundaries of internal elements to simulate approximations for functions $u_{0,1}(p,\varphi)$. We calculate these by solving the Cauchy problem for eqn. (6) for values of variable τ outside the internal elements, and then we prolong it inside Ω_1 using traditional boundary conditions of the continuity of the potential and the normal (with respect to the approximate rectangle boundary) component of the induction.

In the first example the no homogeneous structure has the general characteristic of background $\varepsilon_0(x,y)=1$, and two different internal elements have $\varepsilon_1(x,y)=2$, $\varepsilon_2(x,y)=3$. Results of the structure image restoration in the first experiment are presented in Figure 11: graph (a) – exact distribution; graphs (b), (c), (d) – reconstruction of the structure image by *GR*–algorithm for the number of discrete points $n=20,\ 40,\ 100$, respectively.

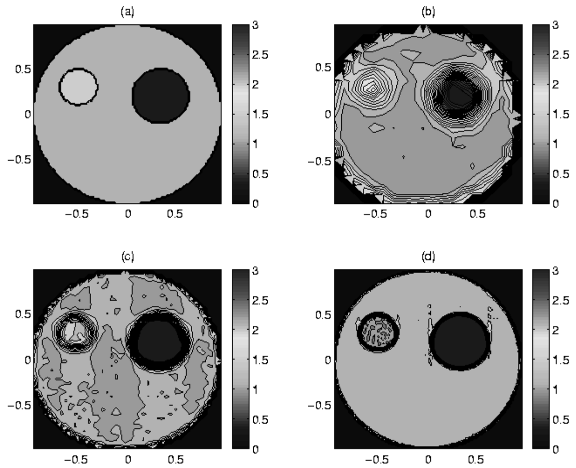

Figure 11: Reconstruction of three component structure obtained using exact input data by *GR*-algorithm.

A more difficult case for the reconstruction corresponds to the greater scale of values $\hat{\varepsilon} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, when the post-processing (projection) is required even for the pre-reconstruction that used exact data. In the second example we use exact values in discrete points for the case $\varepsilon_0(x, y) = 2$, $\varepsilon_1(x, y) = 1$, $\varepsilon_2(x, y) = 70$. In Figure 12 there are presented reconstructions of the structure image by *GR*–algorithm: graph (a) – exact distribution; graph (b) – reconstruction without post-processing, graph (c) - reconstruction with the absolute criterion projection; (d) – reconstructions with the relative criterion projection.

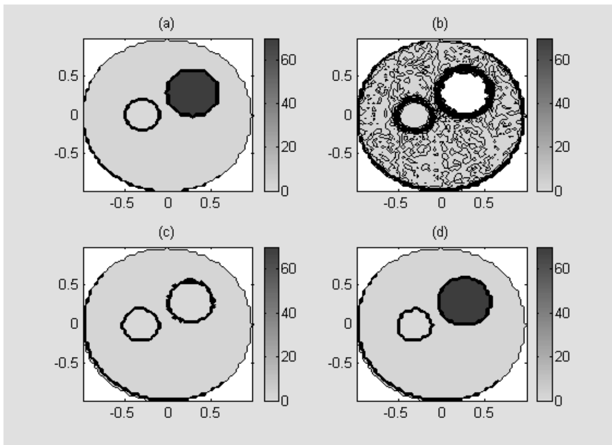

Figure 12. Reconstruction of three component structure by *GR*-algorithm for $\varepsilon_0 = 2$, $\varepsilon_1 = 1$, $\varepsilon_2 = 70$.

The third presented numerical experiment corresponds to the reconstruction of the structure, using simulated noised input data, i.e. values of a function $\overline{v}(p, \varphi) = v(p, \varphi)(1 + \delta(p, \varphi))$, where $\delta(p, \varphi)$ is the randomized function with estimation: $\| \delta(p, \varphi) \|_{C[\Omega]} \le \delta$. Results of the regularized reconstruction for *n*=31, $\delta = 0.05$ are presented in Figure 13: graph (a) - exact $g(x, y)$; graph (b) - reconstruction with noised $\overline{v}(p, \varphi)$ without regularization; graph (c) - reconstruction with noised $\overline{v}(p, \varphi)$ with RS only; graph (d) - reconstruction with noised $\overline{v}(p, \varphi)$ by RS and the absolute criterion projection of pre-reconstructed image.

10. CONCLUSIONS AND ACKNOWLEDGEMENT

The new approach and *GR*-method for the solution of the direct boundary value problems for the elliptic and parabolic differential equations as well as for one inverse coefficient problem are proposed. The approximation properties of the constructed algorithms are justified by numerical experiments. The developed approach can be applied to more general, including multidimensional, problems. The author acknowledges VIEP BUAP for the partial support of the investigation.

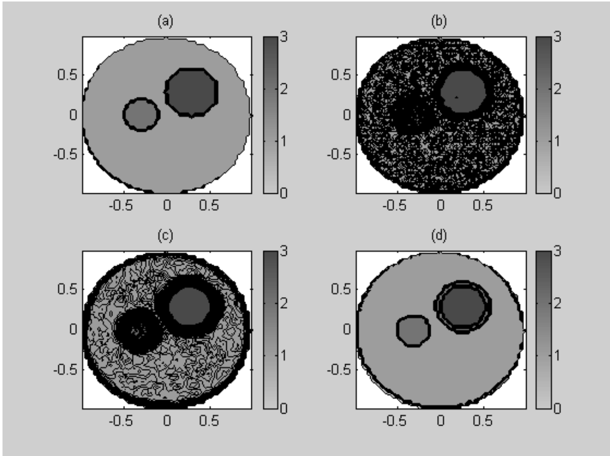

Figure 13. Reconstruction of three component structure using noised data by regularised *GR*-algorithm for $\varepsilon_0 = 1$, $\varepsilon_1 = 2$, $\varepsilon_2 = 3$.

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